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On Normed Linear spaces Which Are Proximinal in Every Superspace

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A linear subspace G of a normed linear space E is said to be *proximinal* if every $x \in E$ has at least one element of best approximation $g_0 \in G$ (i.e., such that $\| x - g_0 \| = \inf_{g \in G} \| x - g \|$).

It is known (see e.g. [7, p. 100, Corollary 2.1′]) that if G is a reflexive Banach space, then G is proximinal in every superspace E (i.e., in every normed linear space E containing G as a subspace). Recently Pollul has proved (see [6, 3]) that the converse is also true, namely, each nonreflexive Banach space can be embedded isometrically as a nonproximinal hyperplane in another Banach space. However, his proof has used the deep theorem of James [4] (for which only difficult proofs are known today) that on every nonreflexive Banach space G there exists a continuous linear functional which does not attain its supremum on the unit cell of G. In the present paper we want to propose a different and more elementary proof, which does not make use of James's theorem. For simplicity we shall assume that the scalars are real; the result also holds for complex scalars, with obvious changes in the proof.

A relevant result related to this problem was obtained by Klee, who has proved [5, Theorem 1] that if E is a nonreflexive Banach space, then for every (closed) hyperplane G in E there exists an equivalent norm on E such that in this new norm G is nonproximinal. (We mention that in [5, Theorem 2], a slightly more general result concerning closed linear subspaces instead of hyperplanes was also given, again for an equivalent norm on E). However, this does not solve the problem, since the equivalent norm on E constructed in [5] induces a different norm on G. We shall prove the result by slightly modifying the construction of [5], so as to obtain an equivalent norm on E which induces a norm on G coinciding with the initial norm.

THEOREM. *A normed linear space G is proximinal in every superspace E if and only if G is a reflexive Banach space.*

Proof. The sufficiency part was mentioned above. Conversely, observe that a normed linear space G which is proximinal in every superspace must be complete, i.e., a Banach space, since every noncomplete normed linear space is nonproximinal in its completion. Thus, it remains only to prove that if G is a nonreflexive Banach space, then there exists a superspace E of G such that G is nonproximinal in E.

Let $E = G \times R$, where R denotes the field of real numbers, or, in other words, let E be an arbitrary Banach space containing G as a hyperplane. Then, since G is nonreflexive, by a theorem of Šmulyan (see e.g. [1, p. 433, Theorem 2]) there exists a decreasing sequence $C_1 \supset C_2 \supset \cdots$ of bounded closed convex subsets of G such that $\bigcap_{n=1}^{\infty} C_n = \varnothing$ (=the empty set). We may assume, without loss of generality, that $C_1 \subset C_0$, where

$$C_0 = \{y \in G \mid \| y \| \leqslant 1\}. \tag{1}$$

Let $C_{-n} = -C_n (n = 1, 2, \ldots)$ and let $x \in E$ be such that $\| x \| < 2$ and $\operatorname{dist}(x, G) > 1$. Set

$$C = \bigcup_{-\infty<n<\infty} [C_n + (\operatorname{sign} n)(1 - 1/2^{|n|})\, x] \tag{2}$$

and let $B = \langle co \rangle\, C$, the closed convex hull of C. Then, similarly to the argument of [5], it follows that the Minkowsky functional

$$\| x \|_1 = \inf_{\substack{\lambda>0 \\ x\in\lambda B}} \lambda \qquad (x \in E) \tag{3}$$

of B is an equivalent norm on E, in which G is nonproximinal. Thus, it remains to prove that $\| y \|_1 = \| y \|$ for all $y \in G$, or, equivalently, that

$$B \cap G = C_0 . \tag{4}$$

The inclusion $C_0 \subset B \cap G$ is obvious by (1) and (2). In order to prove the opposite inclusion, consider the closed convex set

$$A = C_0 + \{\lambda x \mid -\infty < \lambda < \infty\}. \tag{5}$$

Since $C_n \subset C_0$ and $C_{-n} = -C_n \subset -C_0 = C_0$ $(n = 1, 2, \ldots)$, we have

$$C_n + (\operatorname{sign} n)(1 - 1/2^{|n|})\, x \subset A \qquad (-\infty < n < \infty)$$

whence, by (2), $B = \langle co \rangle\, C \subset A$. However, $A \cap G \subset C_0$, since for any $z = y + \lambda x \in A \cap G$ (where $y \in C_0$) we have $\lambda x = z - y \in G - C_0 \subset G$, whence $\lambda = 0$ (because $\operatorname{dist}(x, G) > 1$) and hence $z = y \in C_0$. Consequently,

$$B \cap G \subset A \cap G \subset C_0 ,$$

and thus we have (4), which completes the proof.

Remark 1. The difference between the above construction and that of [5] consists in the fact that in [5] the set C_0 defined by (1) is replaced by $C_0' = \{x \in E \mid \| x \| \leqslant 1\}$, the unit cell of the wole space E. This ensures that $C_0' \subset B$, but makes possible also the situation when $B \cap G \neq C_0' \cap G = C_0$ (which can happen when there exists no linear projection of norm 1 of E onto G). In the above construction we have in general only $\alpha_0 C_0' \subset B$ for some α_0 with $0 < \alpha_0 \leqslant 1$ (this follows from $\alpha_0 C_0' \subset \langle co \rangle \{-C_1 - (1/2), C_0, C_1 + (1/2)\}$, which holds because G is a hyperplane in E and $x \in E \backslash G$), but (4) is ensured.

Remark 2. The above theorem disproves the claim made in [2, p.119], that any conjugate Banach space $G = F^*$ is proximinal in every superspace E. The error in the proof of [2] consists in the assertion that for $x \in E \backslash F^*$ the closed cells S with center x and radius $\operatorname{dist}(x, F^*) + (1/n)$ intersect F^* in $\sigma(F^*, F)$-compact sets; in fact, it is easy to give counter-examples even with this intersection containing some cell of F^*.

The above claim about conjugate spaces was used in [2] to derive the following statement [2, p. 118, Proposition 5, (8)]: If E is a normed linear space and G_1, G_2 are subspaces of E such that $G_1 \supset G_2$, G_2 is proximinal in E and G_1/G_2 is a conjugate Banach space F^*, then G_1 is proximinal in E. The following is a counter-example: Let $E_0 = l^1$, $G_1 = \{x = \{\xi_n\} \in l^1 \mid \xi_1 = 0\}$, $G_2 = [e_2] =$ the line $\{0, \lambda, 0, 0,...\} \mid -\infty < \lambda < \infty\}$, and let E be the space $E_0 = l^1$ endowed with an equivalent norm for which the hyperplane G_1 is not proximinal, but which induces the same norm on G_1 as E_0. Then G_2 is proximinal in E (since $\dim G_2 = 1$) and

$$G_1/G_2 \equiv \{x = \{\xi_n\} \in l^1 \mid \xi_1 = \xi_2 = 0\} \equiv l^1 \equiv c_0^*,$$

where $\equiv$ means linear isometry, but G_1 is not proximinal in E.

Note. Wulbert has observed that our last example can be replaced by the trivial example of $G_1 =$ any nonproximinal conjugate space in a Banach space E and $G_2 = \{0\}$.

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